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ON SUBSONIC PROPAGATION OF THE EDGE OF SHEAR DISPLACEMENT WITH FRICTION ALONG A BOUNDARY SEPARATING ELASTIC MATERIALS*

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Analysis of the stress and velocity fields at the edge of a cut (and in particular of a transverse shear crack), gives rise to the problem studied here, of the asymptotics of dynamic solutions near a moving point represented by the point at which the boundary conditions at the inner boundary of a piecewise homogeneous, linearly ealstic body, change. The sets of boundary conditions of slippage type with dry and viscous friction (or without friction) and adherence with (or without) restrictions on the magnitude of shear stress of static friction type, are considered. It is shown that homogeneous solutions of the problems with similar mixed boundary conditions have nonoscillatory character and the singularity index depends on the velocity, friction coefficients and other material parameters. Velocity ranges are noted within which the stresses are singular or continuous. Results are given of the computations of the angular dependence of the shear stress at different rates of motion of the transverse shear crack edge, and the effect of changing the direction of the maximum stress is studied.

The nature of the singularity at the crack tips of normal separation and longitudinal shear type with smooth contact, propagating along the interface boundary was studied in /1/, where a review and bibliography were also given.

1. Formulation of the problem. Let $\Omega_1, \Omega_2 \in R^3$ denote the regions occupied by homogeneous elastic bodies 1 and 2, $S = \bar{\Omega}_1 \cap \bar{\Omega}_2$ be the general boundary (surface) divided by the curve $\Gamma(t)$ into the regions $S_1(t)$ and $S_2(t)$ (t is time), Q be the point belonging to the regular segment of the curve Γ and surface S, let the velocity $\mathbf{c}$ and acceleration $(d/dt)\mathbf{c}$ of this point relative to the medium be bounded functions of time, and let $P \perp \Gamma$ denote the plane containing Q. We also assume that some correct initial formulation of the problem of the linear dynamic theory of elasticity exists for the system of bodies in question and different conditions of contact are postulated on S_1 and S_2 so that Γ is a singular line.

We construct the asymptotics of the stress (and velocity) field as $r \to 0$ (r is the distance from Q in plane P) at a fixed moment of time, without solving the problem completely. To do this we transfer the problem in the usual manner, to the moving Cartesian coordinate system X_n ($n = 1, 2, 3$) with the origin at the point Q, so that the axes X_1 and X_2 lie in the plane P and axis X_1 belongs to the plane tangent to S at the point Q, $X_3 \perp P$. Following "the microscope principle" /2/ we extend the coordinates by means of the transformation

$$x_1 = \varepsilon X_1,\ x_2 = \varepsilon X_2,\ x_3 = \delta X_3$$

and pass, in the problem A_ε written in coordinates x_n, formally to the limit as $\varepsilon \to 0$, $\delta \to 0$, $\varepsilon/\delta \to 0$. This yields the limiting problem A_0 containing a truncated and disintegrating system of Lamé equations describing plane steady motions of an elastic medium in the $x = x_1$, $y = x_2$ coordinate system moving with the instantaneous velocity $\mathbf{c}(t) = (c, 0)$. The geometry of the small neighborhood of the point Q expanded to infinity represents a plane unbounded region with a rectilinear boundary separating the two bodies, and a point on this boundary at which the boundary conditions change. Thus the problem of investigating the field singularities qualitatively reduces to solving a set of limit (canonical, fundamental) problems /2—5/.

We shall restrict ourselves to the study of subsonic mode and fix only such instants of time, at which the wave fronts carrying infinite discontinuities in the time derivatives of the displacements of up to the second order, do not intersect the small neighborhood of the point Q. The latter is generally necessary for the correctness of the passage to the limit mentioned above, although not compulsory: an example in /2/, p.133, shows that diffraction of the shock waves does not influence the nature of the singularity, it is only the dependence of the intensity coefficient on time that leads to formation of a break. If in view of this we exclude from our considerations the characteristic values of the velocity c (the velocities of the surface waves) in $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Then it becomes physically certain that the solution to

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the limit problem separates out the principal part of the asymptotics of the solution to the problem A_ε. From the mathematical point of view the problem A_0 is obtained, in particular, by replacing the perturbed, hyperbolic type operator, by an elliptic one representing the principal part of the first operator relative to the asymptotics sought. The operator contains all higher order derivatives in the variables in question. In addition, the analogs of the problem A_0 with single type boundary conditions have a discrete spectrum with simple eigenvalues and we can therefore speak of a regularly perturbed problem /6/.

We solve the canonical problems using the functions in complex form, in order to seek, in addition to the homogeneous solutions, also the inhomogeneous ones. We impose on the function sought, namely on the stresses σ_{ml}^j and mass velocities u_m^j at $y=0, x>0$ and $x<0$, the following idealized conditions and certain of their combinations:

1°. Total contact (adherence)

$$[u_m]=0,\ [\sigma_{m2}]=0$$

2°. Total contact with the magnitude of the shear stress restricted

$$[u_m]=0,\ [\sigma_{m2}]=0,\ |\sigma_{12}^j|+k^\circ\sigma_{22}^j\leqslant 0$$

3°. Slippage with dry friction

$$[u_2]=[\sigma_{22}]=0,\quad |\sigma_{12}^j|+k\sigma_{22}^j=0,\quad [u_1]\sigma_{12}^j>0$$

4°. Slippage without friction

$$[u_2]=[\sigma_{22}]=0,\ \sigma_{12}^j=0,\ [u_1]\neq 0$$

5°. Slippage with viscous firction

$$[u_2]=[\sigma_{22}]=0,\ \sigma_{12}^j=\eta\,[u_1],\ [u_1]\neq 0$$

In addition to 2°–5° we have the condition of edge closure indicating the absence of attraction forces between the edges

$$\sigma_{22}^j\leqslant 0 \tag{1.1}$$

The energetic condition supplementing the conditions 1°–5° (in particular, this condition does not permit the creation of mechanical energy at a point) has the form

$$0\leqslant F<\infty \tag{1.2}$$

Here the square brackets denote a jump in the value of the parameter during the passage from the upper to the lower edge of the crack $[f]=f^1(x,0)-f^2(x,0)$, k°, k, η are coefficients of friction, F is the energy flux into the point $\mathbf{x}=0$, $j, m, l=1, 2$; and the superscript accompanying the function denotes the medium. We note that condition (1.2) was used in the proof of a certain uniqueness theorem /7/ and in discussing the problems of propagation of shear cracks (homogeneous material) /8/. Conditions (1.1) and (1.2) limit the admissible singularities of the solution and single out from the initial problem a unique solution out of the set of solutions generated by the basic conditions.

In the stationary problem of the dynamic theory of elasticity (plane deformation, subsonic mode) it is expedient to represent the functions sought in terms of the analytic functions χ_m^j resembling the representations of L.A. Galin /9/

$$\sigma_{12}^j=R_j^{-1}\,\mathrm{Im}\,\{\beta_{1j}\beta_{2j}\chi_1^j(z_{1j})-\beta_j^2\chi_1^j(z_{2j})-\beta_j\beta_{1j}[\chi_2^j(z_{1j})-\chi_2^j(z_{2j})]\} \tag{1.3}$$

$$\sigma_{11}^j=R_j^{-1}\,\mathrm{Re}\,\{-\alpha_j\beta_{2j}\chi_1^j(z_{1j})+\beta_j\beta_{2j}\chi_1^j(z_{2j})+\alpha_j\beta_j\chi_2^j(z_{1j})-\beta_{1j}\beta_{2j}\chi_2^j(z_{2j})\}$$

$$\sigma_{22}^j=R_j^{-1}\,\mathrm{Re}\,\{\beta_j\beta_{2j}[\chi_1^j(z_{1j})-\chi_1^j(z_{2j})]-\beta_j^2\chi_2^j(z_{1j})+\beta_{1j}\beta_{2j}\chi_2^j(z_{2j})\}$$

$$u_1^j=\frac{c}{2\mu_jR_j}\,\mathrm{Re}\,\{\beta_{2j}\chi_1^j(z_{1j})-\beta_j\beta_{2j}\chi_1^j(z_{2j})-\beta_j\chi_2^j(z_{1j})+\beta_{1j}\beta_{2j}\chi_2^j(z_{2j})\}$$

$$u_2^j=\frac{c}{2\mu_jR_j}\,\mathrm{Im}\,\{-\beta_{1j}\beta_{2j}\chi_1^j(z_{1j})+\beta_j\chi_1^j(z_{2j})+\beta_j\beta_{1j}\chi_2^j(z_{1j})-\beta_{1j}\chi_2^j(z_{2j})\}$$

$$\beta_{mj}=\sqrt{1-c^2/c_{mj}^2},\quad \beta_j={}^1/_2(1+\beta_{2j})^2,\quad \alpha_j=1+\beta_{1j}^2-\beta_j$$

$$R_j=\beta_{1j}\beta_{2j}-\beta_j^2,\ z_{mj}=x+i\beta_{mj}y,\ m,\ j=1,\ 2$$

Here c_{1j} and c_{2j} are the velocities of the expansion and shear volume waves, μ_j is the shear modulus, R_j are the Rayleigh functions, $(c_{R_j}>0$ are the roots of the equations $R_j(c)=0)$, and the last subscript a-companying the coefficients denotes the medium. The representations (1.3) determine the velocity field with the accuracy of up to a constant (with the accuracy of up to the motion of the system as a rigid body).

At the boundary $z_{mj}=x$ and from (1.3) we obtain

$$\sigma_{12}{}^j = \operatorname{Im}\chi_1{}^j(x),\ u_1{}^j = c\operatorname{Re}\{b_{2j}\chi_1{}^j(x) + a_j\chi_2{}^j(x)\} \tag{1.4}$$
$$\sigma_{22}{}^j = \operatorname{Re}\chi_2{}^j(x),\ u_2{}^j = -c\operatorname{Im}\{a_j\chi_1{}^j(x) + b_{1j}\chi_2{}^j(x)\}$$
$$a_j = \frac{\beta_{1j}\beta_{2j} - \beta_j}{2\mu_j R_j},\quad b_{mj} = \frac{\beta_{mj}(1-\beta_j)}{2\mu_j R_j},\quad j, m = 1, 2$$

Before passing to the concrete analysis, we derive some general corollaries. The following estimates follow from (1.2):

$$|\chi_m{}^j| < \text{const}\cdot|z|^{-1/2},\ |z|\to 0\ (j, m = 1, 2;\ z = z_{mj}) \tag{1.5}$$

Equations $[\sigma_{12}] = [\sigma_{22}] = 0$, characteristic for all conditions 1°– 5° will be satisfied if

$$\bar{\chi}_1{}^1(\bar{z}) = -\chi_1{}^2(z),\quad \bar{\chi}_2{}^1(\bar{z}) = \chi_2{}^2(z),\quad \operatorname{Im} z < 0 \tag{1.6}$$

Conversely, conditions $[\sigma_{12}] = [\sigma_{22}] = 0$ and (1.5) yield (1.6) with the accuracy of up to the nonessential (not affecting the state of stress) entire functions. Taking into account (1.4) and (1.6) we write the formulas for the velocity jumps at $y = 0$, which shall be required later

$$[u_1] = c\,(q\operatorname{Re}\chi_1{}^1 + d\operatorname{Re}\chi_2{}^1),\ [u_2] = -c\,(d\operatorname{Im}\chi_1{}^1 + p\operatorname{Im}\chi_2{}^1) \tag{1.7}$$
$$d = a_1 - a_2,\ p = b_{11} + b_{12},\ q = b_{21} + b_{22}$$

The condition $[u_2] = 0$ common to 1°– 5°, gives rise to a relation connnecting the functions $\chi_1{}^1$ and $\chi_2{}^1$

$$\chi_2{}^1(z) = -\frac{d}{p}\chi_1{}^1(z) + P_e(z),\quad \operatorname{Im} z > 0 \tag{1.8}$$

where P_e denotes an entire function of the form

$$P_e(z) = \sum_0^\infty e_n z^n,\quad \operatorname{Im} e_n = 0 \tag{1.9}$$

Owing to (1.6) and (1.8), the problem can be reduced to the Hilbert problem with discontinuous coefficients for a single analytic function $\chi_1{}^1$ /10/. The problem is obtained by substituting (1.4), (1.6)–(1.8) into the still imperfect combinations of the conditions 1°– 5°. Such a splitting of the problem ensures that the solutions are monotonous (nonoscillatory) in contrast to the combinations of the boundary conditions of the separation - complete contact-type /1/.

2. Slippage with dry friction - adherence. Let us assume without loss of generality (the sign of c is not discussed) that conditions 1° or 2° hold to the right of the point $x = 0$ at the interface boundary, and 3° to the left (problems 3°– 1° and 3°– 2°). The boundare conditions for $\chi_1{}^1$ are

$$\operatorname{Re}\chi_1{}^1 = -\frac{d}{p}P_\sigma(x),\quad x > 0\quad \left(P_\sigma = -\frac{pq}{S}P_e\right) \tag{2.1}$$
$$\operatorname{sgn}[u_1]\operatorname{Im}\chi_1{}^1 - k\frac{d}{p}\operatorname{Re}\chi_1{}^1 = \frac{kS}{pq}P_\sigma(x)\quad (S = d^2 - pq)$$

It is natural to assume that the sign of the velocity $[u_1]$ of the crack appearing in (2.1) does not change in the small neighborhood of the point at which slippage commences, the neighborhood belonging to the physical space and extended afterwards to infinity. We shall determine this sign, as well as the signs of the remaining quantities, from the sign of the principal term of the asymptotic. As usual, we seek the complete solution of the problem (2.1) in the form of a sum of the general solution of the corresponding problem with homogeneous conditions, and a particular solution of the inhomogeneous problem, bearing in mind (1.5) /10/. The final results are

$$\chi_1{}^1 = iz^\lambda P_N(z) - \left(\frac{d}{q} \pm ik\right)P_\sigma(z) \tag{2.2}$$
$$\chi_2{}^1 = -\frac{d}{p}iz^\lambda P_N(z) + \left(1 \pm i\frac{kd}{p}\right)P_\sigma$$
$$\lambda = \lambda_0 \equiv \pm\operatorname{arctg}(p/(kd))\ \text{(problem 3°–1°)} \tag{2.3}$$
$$\lambda = \lambda_0,\ \text{if}\ \lambda_0 > 0;\ \lambda = 1 + \lambda_0,\ \text{if}\ \lambda_0 < 0\ \text{(problem 3°–2°} \tag{2.4}$$

In (2.2)–(2.4) the upper or lower signs are chosen according to whether $[u_1] > 0$ or $[u_1] < 0$ when $x < 0$. To make the function z^λ uniform, we made a cut along the positive part of the real axis. The condition $(1)^\lambda = 1$ fixes the branch of z^λ at the upper edge of the cut. The eigenvalue λ in (2.3) lies in the interval $-1/2 < \lambda < 1/2$ $(k \neq 0)$, and in (2.4) in the interval

$0 < \lambda < 1$ ($k \neq 0$, $p \neq 0$). In the problem $3^\circ - 2^\circ$ λ is positive, since the third of the conditions 2° must be satisfied.

In contrast to the calssical formulation of the Hilbert problem /10/, no restrictions are imposed here on the behavior of the functions at infinity, and the solution therefore contains the product of the canonical solution and a polynomial of infinite order. The real unknown constants N_n, σ_n ($n = 0, 1, 2, \ldots$) are found from the external boundary conditions of the initial, plane, stationary problem, which were neglected in the present formulation. When more general problems are considered, namely the non-stationary and (or) three-dimensional problems, only the sum of first principal nonzero terms of the asymptotics of the general solution of the inhomogeneous problem for χ_1^1 ($n = 0$) will be meaningful, since the contribution of the neglected part in the equations and boundary conditions will be small only when compared with those terms.

Let us write the first two terms of the asymptotics of the functions at the boundary as $x \to 0$, using (1.4), (1.7) and (2.2)

$$\sigma_{12}{}^j \sim N_0 x^\lambda \mp k\sigma_0, \quad \sigma_{22}{}^j \sim \sigma_0, \quad u_1{}^j = 0 \tag{2.5}$$
$$u_2{}^j \sim c\,(db_{11}/p - a_1)\, N_0 x^\lambda, \quad x > 0$$
$$\sigma_{12}{}^j \sim N_0 \cos(\pi\lambda)\,|x|^\lambda \mp k\sigma_0, \quad \sigma_{22}{}^j \sim \frac{d}{p} N_0 \sin(\pi\lambda)\,|x|^\lambda + \sigma_0$$
$$[u_1] \sim \frac{cS}{p} N_0 \sin(\pi\lambda)\,|x|^\lambda,$$
$$u_2{}^j \sim \frac{c}{p}(db_{11} - pa_1) N_0 \cos(\pi\lambda)\,|x|^\lambda, \quad x < 0$$

We have deleted from the right-hand sides of (2.5) terms of order $O(x)$ in the stresses, and $O(1)$ in velocities. Next we check the supplementary conditions, and since the solution (2.2) – (2.5) contains, in fact, the recursive inversion, we avoid it by showing explicitly the parameter limits within which one or another type of solution is valid. To do this, we make the following assumptions A and B concerning the sign of $[u_1]$.

A. $[u_1] > 0$. Moreover, in the problem $3^\circ - 2^\circ$ we must observe the condition (1.1) for $|x| < \infty$, and take into account the fact that $\lambda > 0$; in the problem $3^\circ - 1^\circ$ we observe the condition (1.1) for $x < 0$ and remember that the sign of λ can vary. From (2.4), (2.5) it follows that the inequality in 2° is valid if $k < k^\circ$, which is usually true. We arrive at the following systems of inequalities:

$$\sigma_0 < 0, \; cpSN_0 > 0 \; (\text{problem } 3^\circ\text{—}2^\circ) \tag{2.6}$$
$$pd < 0, \; cpSN_0 > 0, \; \sigma_0 < 0 \; (\text{problem } 3^\circ - 1^\circ, \lambda > 0) \tag{2.7}$$
$$pd > 0, \; cpS < 0, \; N_0 > 0 \; (\text{problem } 3^\circ\text{—}1^\circ, \lambda < 0) \tag{2.8}$$

The conditions (2.6)–(2.8) specify a range of velocities c (and the signs of the coefficients N_0 and σ_0) in which the upper indices should be taken in (2.2)–(2.5).

B. $[u_1] < 0$. Analogously to the previous case we obtain the following systems of inequalities:

$$\sigma_0 < 0, \; cpSN_0 < 0 \; (\text{problem } 3^\circ\text{—}2^\circ) \tag{2.9}$$
$$\sigma_0 < 0, \; pd > 0, \; cpSN_0 < 0 \; (\text{problem } 3^\circ\text{—}1^\circ, \lambda > 0) \tag{2.10}$$
$$pd < 0, \; cpS < 0, \; N_0 < 0 \; (\text{problem } 3^\circ\text{—}1^\circ, \lambda < 0) \tag{2.11}$$

Lower indices must be taken in conditions (2.9)–(2.11) and solution (2.2)–(2.5). From (2.8) and (2.11) it follows that a singular solution of the problem of propagation of shear crack (problem $3^\circ - 1^\circ$) exists within the range of velocities c defined by the inequalities

$$cpS < 0 \tag{2.12}$$

otherwise ($cpS > 0$) the solution is continuous at the point $x = 0$. Analysis of (2.12) will be given below.

Let us write the formulas for the principal variable of the part of the field asymptotics within the regions, in the polar coordinate system $z_{mj} = r_{mj}e^{i\theta_{mj}}$ ($r_{mj} > 0$, $0 \leqslant (-1)^{j+1}\theta_{mj} \leqslant \pi$) obtained from (1.3) and (2.2)

$$\sigma_{11}{}^j \sim N_0\,(\alpha_j A_{1j} - \beta_{2j}A_{2j}), \quad \sigma_{12}{}^j \sim N_0\,(\beta_{1j}B_{1j} - \beta_j B_{2j}) \tag{2.13}$$
$$\sigma_{22}{}^j \sim N_0\,(\beta_{2j}B_{2j} - \beta_j B_{1j}), \quad (u_1{}^j u_2{}^j) \sim \frac{cN_0}{2\mu_j}(\beta_{2j}A_{2j} - A_{1j}, B_{2j} - \beta_{1j}B_{1j})$$
$$A_{mj} = \varkappa_{mj} r_{mj}^\lambda \sin(\lambda\theta_{mj}), \quad B_{mj} = \varkappa_{mj} r_{mj}^\lambda \cos(\lambda\theta_{mj}) \quad (j, m = 1, 2)$$
$$\varkappa_{1j} = R_j^{-1}(\beta_{2j} + (-1)^{j+1}\beta_j d/p), \quad \varkappa_{2j} = R_j^{-1}(\beta_j + (-1)^{j+1}\beta_{1j} d/p)$$

3. Slippage without friction - adherence (problems $4^\circ-1^\circ, 4^\circ-2^\circ$). As in Sect.2, we arrive at the solution (2.2), (2.5), (2.13), in which we must put $\lambda = -1/2$ (problem $4^\circ-1^\circ$) and $\lambda = 1/2$ (problem $4^\circ-2^\circ$). In the combination $4^\circ-1^\circ$ we have a nonzero flow of energy into the point $x = 0$ $(N_0 \neq 0)$. To calculate it, we rewrite the asymptotics (2.5) as follows $(k = 0, \lambda = -1/2)$

$$\sigma_{12}{}^j \sim N_0 x^{-1/2}, \ \sigma_{22}{}^j \sim \sigma_0, \ u_1{}^j = 0, \ x > 0 \tag{3.1}$$

$$\sigma_{12}{}^j = 0, \quad \sigma_{22}{}^j \sim -\frac{dN_0}{p\,|x|^{1/2}} + \sigma_0, \quad [u_1] \sim -\frac{cSN_0}{p\,|x|^{1/2}}, \quad x < 0$$

The restriction $pdN_0 > 0$, so far unique, follows from (1.1).

The essential difference between (3.1) and theanalogous expressions for the homogeneous material /8,11/ is represented by the singularity $\sigma_{22}{}^j(x, 0)$ at $x < 0$, which vanishes, when the difference between the materials is removed $(d \to 0)$. All distributions (2.13), (3.1) now transform into the corresponding formulas for the shear crack in a homogeneous material /11/.

To compute F we use of the simplest methods of calculating this quantity with help of the intensity factors $\sigma_{m2}{}^j$ and $[u_m{}^j]$ [11], and obtain the formula

$$F = -\pi cSN_0{}^2/(2p) \tag{3.2}$$

Condition (1.2) and (3.2) together yield the inequality (2.12). When $cpS > 0$, we must put $N_0 = 0$. This removes the singularity, and the field asymptotics will now contain terms of order $O(1)$ and $O(r^{1/2})$. Identical results are obtained by passing in (2.2)–(2.5), (2.13) to the limit as $k \to 0$, and retaining the conditions (2.6)–(2.11). In this sense the problem will not be degenerate when $k = 0$. From the point of view of the theory of brittle fracture however, it is degenerate: when $k \neq 0$ we have $F = 0$, while when $k = 0$ and condition (2.12) holds, we have $F \neq 0$. Results of Sect.4 impy that when the materials are identical, then the cases $k = 0$ and $k \neq 0$ do not differ greatly from each other even in the sense of the theory of fracture.

Let us analyze the condition (2.12) for the case $c_{R1} < c_{R2} < c_{21} \leqslant c_{22}$ (materials with similar properties). In this case the following distribution of the unique positive roots of the equations $S(c) = 0$ and $p(c) = 0$ is possible, the roots denoted by c_S and c_p:

$$c_{R1} < c_p < c_S < c_{R2}, \quad c_{R1} < c_p < c_{R2} < c_S < c_{21}$$

It can be shown that when $c_{R2} \to c_{R1}, c_{22} \to c_{21}, \mu_2 \to \mu_1$, then the root c_S determining the velocity of the Stoneley wave /12/ tends to c_{21}. In this, and the other case, the inequality (2.12) is true when

$$0 \leqslant c < c_p, \ c_S < c < c_{21}, \ -c_S < c < -c_p, \ |c| \neq c_{Rj}$$

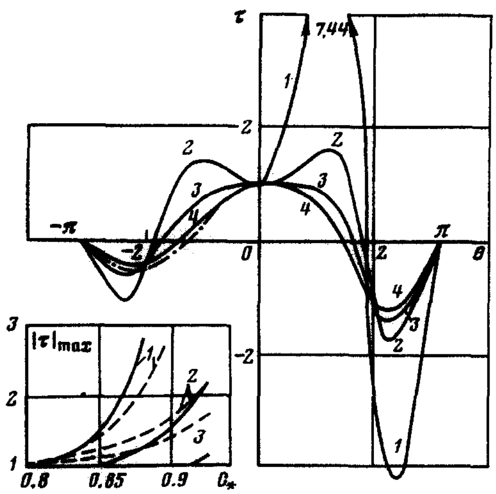

Fig.1

Let us now consider the case when the materials have sufficiently different properties $c_{R1} < c_{21} < c_{R2} < c_{22}$. The following variants are possible:

a) $c_{21} < c_p < c_S$, then $cpS < 0$ when $0 < c < c_{21}, c \neq c_{R1}$

b) $c_{R1} < c_p < c_{21} < c_S$, then $cpS < 0$ when $0 < c < c_p, c \neq c_{R1}, \quad -c_{21} < c < -c_p$

The angular distribution of shear stress is of interest in connection with potential possibility of crack branching /1/. The Fig.1 shows the plots of the function $\tau(\theta)$ computed according to the formula

$$\tau = \frac{\sigma_{r\theta}{}^{j}(r,\theta)}{\sigma_{r\theta}{}^{j}(r,0)} \sim [{}^1/_2(1+\beta_{1j})C_{1j} - \beta_{2j}C_{2j}]\sin(2\theta) + (\beta_{1j}D_{1j} - \beta_j D_{2j})\cos(2\theta)$$

$$r = (x^2+y^2)^{1/2}, \quad \theta = \operatorname{Arc\,tg}(y/x), \quad (C_{mj}, D_{mj}) = \varkappa_{mj}\frac{r}{r_{mj}}\left(\sin\frac{\theta_{mj}}{2}, \cos\frac{\theta_{mj}}{2}\right)$$

$$\frac{r}{r_{mj}} = \left(\frac{1+\mathrm{tg}^2\theta}{1+\beta_{mj}^2\mathrm{tg}^2\theta}\right)^{1/2}, \quad \theta_{mj} = \begin{cases} \overset{\circ}{\theta}_{mj} \equiv \operatorname{arc\,tg}(\beta_{mj}\,\mathrm{tg}\,\theta), & |\theta| < \pi/2 \\ (-1)^{j+1}\pi + \overset{\circ}{\theta}_{mj}, & \pi/2 < |\theta| < \pi \end{cases}$$

The formula was obtained from (2.13) at $\lambda = -{}^1/_2$ and contains dimensionless parameters $\mu = \mu_1/\mu_2$, $\rho = \rho_2/\rho_1$ (relative density), ν_j (Poisson's ratios) and $c_* = c/c_{21}$. The values of μ equal to 1, (identical materials), 0.75, 0.5 and 0.1 correspond to the curves *1—4*. The remaining parameters for the curves $\tau = \tau(\theta)$ are fixed $\rho = 1$, $\nu_1 = \nu_2 = 0.3$ (which corresponds to $c_{R1}/c_{21} \approx 0.9274$), $c_* = 0.91$. By a dot-and-dash line is shown one the symmetric branches $\tau(\theta)$, for a homogeneous medium in a quasistatic case ($c_* = 0.1$). Solid lines in the lower left part of the Fig.1 show the maximum values of $|\tau|$ in the interval $0 < \theta < \pi/2$, and dashed lines correspond to the interval $\pi/2 < \theta < \pi$. The table gives extremal values of τ denoted by τ_n ($n = 1, 2, 3, 4$) and attained, respectively, at the points θ_n lying in the intervals $0 < \theta_1 < \pi/2$, $\pi/2 < \theta_2 < \pi$, $-\pi/2 < \theta_3 < 0$, $-\pi < \theta_4 < -\pi/2$. A gap indicates that the maximum is attained at the boundary $\theta = 0$ and is equal to unity. We have here $\nu_j = 0.3$, and the condition that the medium 1 is not the high velocity one is observed $c_{m1} \leqslant c_{m2}$, $m = 1, 2$. We note the relation $(\mu\rho)^{1/2} = c_{21}/c_{22}$.

Table 1

	μ=1; ρ=1			μ=0.75; ρ=1			μ=0,2; ρ=5		
	C_*=0.82	0.88	0.91	0.85	0.88	0.925	0.79	0.88	0.91
τ_1	1.12	2.55	7.41	—	1.20	2.24	—	1.07	2.63
τ_2	−1.20	−1.87	−4.27	−1.28	−1.48	−2.20	−1.30	−1.43	−2.26
τ_3	1.12	2.55	7.41	—	1.11	1.80	1.27	4.14	12.32
τ_4	−1.20	−1.87	−4.27	—	—	−1.23	−1.11	−2.52	−6.32

	μ=1; ρ=0,5			μ=1.5; ρ=0.33			μ=0.33; ρ=1.5		
	c_*=0.88	0.91	0.925	0.85	0.88	0.925	0.85	0.88	0.925
τ_1	1.09	1.43	1.89	1.08	1.34	2.44	—	—	—
τ_2	−1.44	−1.74	−2.07	−1.30	−1.54	−2.29	−1.17	−1.28	−1.53

Computations show that the maximum of $|\tau|$ is displaced deeper into the region beginning from the value of velocity equal to $c_* \approx 0.77$ ($\nu_j = 0.3$), and is first attained when $\theta \approx 3\pi/4$. Next ($c_* \approx 0.80$ when $\mu = \rho = 1$, $c_* \approx 0.86$ when $\mu = 0.75$, $c_* \approx 0.92$ when $\mu = 0.5$) other extrema appear exceeding unity in modulo, and when $c_* \to c_{R1}/c_{21}$, then the maximum at the point $\theta = \theta_1$ becomes dominant (see Fig.1 and Table 1). The value of the angle θ_1 is changed from $\approx \pi/4$ to ≈ 1.36 when $c_* = 0.925$ and the angle θ_2 remains nearly equal to $3\pi/4$.

The above results show that asymmetry leads to appreciable reduction in the effect of the stress concentration $\sigma_{r\theta}{}^{j}$ in directions different from the direction in which the cut propagates, and practically vanishes in the case of materials with sufficiently different wave velocities c_{2j}. On the other hand, if the medium 2 is rigid, then

$$\varkappa_{11} = [\beta_{11}(1-\beta_1)]^{-1}, \ \varkappa_{21} = (1-\beta_1)^{-1}$$

and the problem of Sect.2 and 3 lose the resonance when $c = c_{R1}$. When ν_1 is varied, then all changes depend essentially on the change in position of the resonance value of the velocity c. When $k, \eta \ll 1$, we have $\lambda = -{}^1/_2 + O(k)$, $-{}^1/_2 + O(\eta)$ in the singular variants of the problems with friction, and the results for $\tau(\theta)$ will be close to those obtained above.

4. Slippage with viscous friction - adherence (problems 5°– 1°, 5°– 2°).

As in Sect.2, we obtain the solution

$$\chi_1{}^1(z) = iz^\lambda P_N(z) - \frac{d}{q}P_\sigma(z), \ \chi_2{}^1 = \frac{d}{p}iz^\lambda P_N(z) + P_\sigma(z), \quad \operatorname{Im} z > 0 \qquad (4.1)$$

$\lambda = \lambda_0 \equiv \operatorname{arctg}[p/(c\eta S)]$ (problem $5°-1°$)

$$\lambda = \lambda_0, \text{ if } cpS > 0; \lambda = 1 + \lambda_0, \text{ if } cpS < 0 \text{ (problem } 5^\circ - 2^\circ)$$

Formulas (2.13) remain in force and we must put $k = 0$ in (2.5). The singular solution of the problem $5^\circ - 1^\circ$ is realized with (2.12) observed, just as in the cases discussed before.

5. Degenerate cases. If the velocity c coincides in modulo with one of the values c_{R1}, c_{R2}, c_d, c_q, c_p, c_S (c_d, c_q are roots of $d(c) = 0$, $q(c) = 0$), then the problems discussed in Sect.2–4 becomes meaningless. The formula (2.13) implies that the values $|c| = c_{R1}, c_{R2}, c_p$, are resonant, stationary-problems have no solutions and the solutions of the corresponding nonstationary problems ($c = \text{const}$) have no limit as $t \to \infty$ /13/. The physical meaning of the roots c_p and c_q is explained in /14/. They represent the velocities of the characteristic (surface) waves under the condition of smooth contact between two different elastic bodies, and of non-slippage with possible detachment ("comb"-type condition) respectively.

When $|c| = c_d, c_q, c_S$, the degeneracy has a different character. Stationary solutions may exist, although the solutions must be inspected once again since some of the coefficients appearing in the boundary conditions for χ_1^1 and χ_2^1 vanish or become infinite.

Let us consider these cases, assuming without loss of generality that $c_{R1} < c_{R2}$.

1°. $c = c_d$. The root $c = c_d$ can exist in the intervals $0 < c < c_{R1}$ and $c_{R2} < c < c_{21} \leqslant c_{22}$ (provided that the latter intervals exist). This can be shown by inspecting the signs of $d(0)$, $d(c_{21})$ and $d(c)$ with $c \to c_{R1} \pm 0, c_{R2} \pm 0$, and conditions of existence of the roots can be obtained in the form

$$\frac{\mu_2}{\mu_1} \leqslant \frac{1 - 2\nu_2}{1 - 2\nu_1} \; (0 \leqslant c_d < c_{R1}), \; \frac{\mu_2}{\mu_1} > \frac{\beta_2 + \beta_{12}\beta_{22}}{2R_2}\bigg|_{c=c_{21}} \quad (c_{R2} < c_d < c_{21})$$

Moreover, $d(c) \equiv 0$ for any c, provided that the bodies 1 and 2 have the same properties. The physical meaning of the roots c_d is, that at these velocities of the crack edge the piecewise homogeneous bodies exhibit in some respect the same properties, as the homogeneous bodies. Formulas given below show cearly that the solutions at the cut for the homogeneous and piecewise homogeneous bodies show no qualitative difference.

The solutions of the problems $3^\circ-1^\circ$, $3^\circ-2^\circ$ at $d = 0$ have the form

$$\chi_1^1 = iz^{-1/2} P_N(z) \mp ikP_\sigma(z), \quad \chi_2^1(z) = P_\sigma(z), \quad \operatorname{Im} z > 0$$

where $N_0 = 0$ in the problem $3^\circ - 2^\circ$.

A qualitative change in the stresses and velocities at the line of slippage takes place in cases when $N_0 \neq 0$ ($cqN_0 < 0$ correspond to the upper indices and $cqN_0 > 0$ to the lower indices).

$$\sigma_{12}{}^j \sim \mp k\sigma_0, \; \sigma_{22}{}^j \sim \sigma_0, \; [u_1] = -cqN_0 \, |x| \ldots, \; u_2{}^j = 0$$

The stresses and the velocity u_2^j were found to be bounded in this case at the cut, and we see why the problem admitted the eigenvalue $\lambda = -1/2$ with the energy flux $F = (\pi/2)\, cqN_0^2$ ($cq > 0$ represents the condition of existence of solution with $N_0 \neq 0$; when $cq < 0$, $N_0 = 0$). The power generated by the friction forces (now restricted) has become integrable also at $\lambda = -1/2$.

Let us now put $d = 0$ in (4.1) (case of viscous friction). It is interesting to note that this case $\lambda_0 = -\operatorname{arctg}[(c\eta q)^{-1}] > -1/2$ and $F = 0$ when $c = c_d$ (piecewise homogeneous medium) and any $|c| < c_{21}$, $|c| \neq c_{R1}$ (identical materials and $\sigma_{22}{}^j(x = 0)$ ceases to be singular. A singularity with the index $\lambda > -1/2$ ensures the integrability of the force of viscous friction propoertional to the integral of the product $[u_1]\,\sigma_{12}{}^j$ [15]. We recall the approximate character of the friction laws used valid in restricted intervals of the relative slippage velocities and pressures. At large values of $\sigma_{22}{}^j$ (dry friction) and $[u_1]$ (viscous friction) the resistance to shear unit, apparently remain restricted. For this reason the solutions discussed above must be regarded as some intermediate asymptotics in the regions near the singularities where the linearized friction laws can still be used. On the other hand, we can assume, that the approximate relation $\sigma^j_{12} = \tau_0 = \text{const}$ holds on the slippage line quite near the singularity. Then the solution of the corresponding canonical problem will consist of the solution of the homogeneous problem without friction (Sect.3), and the particular solution

$$\chi_1^1 = i\tau_0, \; \chi_2^1 = -ip^{-1}d\tau_0$$

Thus we have obtained a solution to the canonical problem of propagation of an infinitely thin half-strip of plastic flow (Tresca flow condition) along the boundary separating two elastic materials.

2°. $c = c_q$. Considering the boundary value problems of Sect.2–4 once again, we arrive at the solution (2.2), (4.1) in which we must put $P_\sigma(z) \equiv 0$. The constant term $\sigma_0 = 0$ vanishes (we recall here the physical meaning of the root c_q).

3°. $c = c_S$. Here $[u_1]$ vanishes when $x < 0$, and this imples slippage. This contradicts the condition adopted in $3^\circ - 5^\circ$, no solutions exist and c_S is the resonance value of the velocity c.

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